1. Exercises from Sections 1.8-2.1

Problem 1. *We call $f : S \to \mathbb{R}^n$ Holder continuous with exponent λ iff there exists constants $C, \lambda > 0$ such that $|f(x) - f(y)| < C|x - y|^\lambda$ for every $x, y \in S$. Show that f is uniformly continuous.*

Proof.
(1) Fix $\epsilon > 0$.
(2) NTS for any $x, y \in S$, we can force $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$
(3) By Holder continuity, $|f(x) - f(y)| < C|x - y|^\lambda$ for all $x, y \in S$.
(4) Pick $\delta = (\epsilon/C)^{1/\lambda}$, then for any $x, y \in S$ such that $|x - y| < \delta$ we have

$$|f(x) - f(y)| < C|x - y|^\lambda < C((\epsilon/C)^{1/\lambda})^\lambda = \epsilon$$

□

Problem 2. *Show that if $f : S \to \mathbb{R}^m$ is uniformly continuous on S and $\{x_k\}$ is a Cauchy sequence in S, then $\{f(x_k)\}$ is Cauchy. Give an example of a Cauchy sequence $\{x_k\} \subseteq (0, \infty)$ and a continuous function $f : (0, \infty) \to \mathbb{R}$ such that $\{f(x_k)\}$ is not Cauchy.*

Proof. (1) Assume that we can force $|x_k - x_l| < \delta$ by picking $k, l > N$ sufficiently large
(2) Fix $\epsilon > 0$
(3) By uniform continuity, we can force $|f(x) - f(y)| < \epsilon$ for every $x, y \in S$ such that $|x - y| < \delta$
(4) Now $k, l > N$ implies $|x_k - x_l| < \delta$, so $|f(x_k) - f(x_l)| < \epsilon$ whenever $k, l > N$.

To find a counter example in the case that f is only continuous, consider the function $f(x) = 1/x$. This is continuous on $(0, \infty)$ because $|1/x - 1/y| = |x - y|\frac{1}{|xy|} < |x - y|M^2 < \epsilon$ whenever $|x - y| < \epsilon/M^2$, where $1/M = \min\{x, y\}$. Now consider the sequence $x_k = 1/k$. This sequence is Cauchy (show this fact!). Now for any $N > 0$ pick $l > N$, then we notice that $|f(x_k) - f(x_l)| = |k - l| > \epsilon$ by picking $k > l$ sufficiently large. □

Problem 3. *Suppose that f is differentiable on an open interval I and that $f'(x) > 0$ for all $x \in I$ except for finitely many points at which $f'(x) = 0$. Show that f is strictly increasing.*

Proof. Suppose not, then there exist points $a, b \in I$ with $a < b$ such that $f(b) \leq f(a)$. If $f(b) < f(a)$ then immediately obtain a contradiction to the mean value theorem - there would exist some $c \in I$ such that $f'(c) = \frac{f(b)-f(a)}{b-a} < 0$. If $f(b) = f(a)$ then there exists a point $c \in (a, b)$ such that $f'(c) = 0$. If c were a maximum, then $f(c) > f(a) = f(b)$ hence by the intermediate value theorem there exists a point $c' \in (c, b)$ such that $f(c') < 0$ (and similarly if c were a minimum). We can only conclude that $f(c) = f(a) = f(b)$ and $f'(c) = 0$. But now there are points $d \in (a, c)$ and $d' \in (c, b)$ such that $f'(d) = f'(d') = 0$, and by an identical argument these must also satisfy $f(d) = f(d') = f(c) = f(a) = f(b)$. Continuing inductively would give an infinite set $\{c_k\}$ of points at which $f'(c_k) = 0$, contradicting our initial hypothesis. □

Problem 4. *Define $h(x)$ by $h(x) = x^2$ for all $x \in \mathbb{Q}$ and $h(x) = 0$ otherwise. Show that h is differentiable at $x = 0$, even though it is discontinuous everywhere else*

Proof. We showed earlier in the term that a function similar to this one is continuous at zero (but notice it is not differentiable at zero!).

In light of proposition 2.5 we guess that $h'(0) = 0$ and fix $\epsilon > 0$. Consider $x > 0$, then

$$\left|\frac{h(x) - h(0)}{x}\right| < |x^2/x| = |x| < \epsilon \quad \text{as } x \to 0$$

Similarly, if $x < 0$ then:

$$\left|\frac{h(-x) - h(0)}{-x}\right| < |(-x)^2/(-x)| = |x| < \epsilon \quad \text{as } x \to 0$$

We therefore conclude that the derivative $h'(0)$ exists and is equal to zero. $\square$